Fall 2009

1. $x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6, -1$

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x = -3, -1$

solutions that satisfy one but not both are 6, and -3

$6 + (-3) = 3$

So the answer is **E. 3**

2. I choose 0, 1, 2, and 3 as my 4 consecutive integers. Because the 0 cancels whatever term it is multiplied by, all that matters is what the other 2 terms are. The possibilities are:

$1 * 2 = 2$

$1 * 3 = 3$

$2 * 3 = 6$

with the highest as 6 and the lowest as 1.

$6 - 2 = 4$

So the answer is **D. 4**

3. $x * \left(b + \frac{1}{x}\right) = y$ mathematical translation of the problem.

$xb + 1 = y$

$xb = y - 1$

$x = \frac{y-1}{b}$

So the answer is **A.** $\boldsymbol{\frac{y-1}{b}}$

4. $f(x-2) = 22$

$(x-2)^2 - (x-2) + 2 = 22$

$(x-2)^2 - (x-2) - 20 = 0$

$\big((x-2)-5\big)\big((x-2)+4\big) = 0$

$(x-7)(x+2) = 0$

$x = 7, -2$

$7 + (-2) = 5$

So the answer is **E. 5**

5.

$S = 2.5J$

$S + J = 42$

$2.5J + J = 42$

$3.5J = 42$

$J = 12$

$S = 2.5(12)$

$S = 30$

$0.5S + 1.5J$

$0.5(30) + 1.5(12)$

$15 + 18 = 33$

So the answer is **C. 33**

6. Since a and b are perfect squares, a^4 and b^3 have to be a perfect power of 8 and of 6, respectively. So now we have $x^8 + y^6 + c^2 = 2009$, with $x = \sqrt{a}, y = \sqrt{b}$, both being integers.

The only perfect 8th powers less than 2009 are 1 and 256

The perfect 6th powers less than 2009 are 1, 64, and 729

$2009 - x^8 - y^6 = c^2$, and if we replace x^8 and y^8 with their possibilities, we can look for perfect squares.

$2009 - 1 - 1 = 2007$

$2009 - 1 - 64 = 1944$

$2009 - 1 - 729 = 1279$

$2009 - 256 - 1 = 1752$

$2009 - 256 - 64 = 1689$

$2009 - 256 - 729 = 1024$

Of these, only 1024 is a perfect square (32^2)

So $2009 = 2^8 + 3^6 + 32^2$

$x = 2, y = 3, c = 32$

$a = 4, b = 9, c = 32$

$4 + 9 + 32 = 45$

So the answer is **D. 45**

7.

The most obvious numbers that qualify are 111, 222, and so on to 999. There are 9 of these.

Then there are numbers with 0, an even number, and half of that number, such as 210, 420, etc.

There are 4 number combinations(120, 240, 360, 480), and each can go in 4 orders (0 can't be in front, i.e. 120, 210, 201, 102 is one set of 4), for a total of 16 of this form.

The other set is numbers average from the other 2 without repetition or 0. I list them as a, a+b, a+2b, (b>0, a+2b<10) so the middle number is the average of the other 2

123, 135, 147, 159

234, 246, 258

345, 357, 369

456, 468

567, 579

678

789

there are 16 combinations, and each can go in 6 orders (such as 123, 132, 213, 231, 312, 321) for a total of 96 of this type.

$96 + 16 + 9 = 121$

So the answer is **E. 121**

8. We are basically looking for factorizations of 60 into 3 terms, ordered from least to greatest.

1*1*60

1*2*30

1*3*20

1*4*15

1*5*12

1*6*10

2*2*15

2*3*10

2*5*6

3*4*5

for a total of 10

So the answer is **C. 10**

9.

$$\frac{2\sin x}{\cos x - \sin x \tan x}$$

$$\frac{2\sin x}{\cos x - \sin x\left(\frac{\sin x}{\cos x}\right)} \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\frac{2\sin x}{\frac{\cos^2 x}{\cos x} - \frac{\sin^2 x}{\cos x}}$$ $multiply \cos x\ by \frac{\cos x}{\cos x}, combine\ sin's$

$$\frac{2\sin x}{\frac{\cos^2 x - \sin^2 x}{\cos x}}$$

$$2\sin x * \left(\frac{\cos x}{\cos^2 x - \sin^2 x}\right)$$ flip and multiply fraction.

$$\frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}$$ combine

$$\frac{\sin 2x}{\cos 2x}$$ double angle identities

$\tan 2x$

So the answer is **A. tan 2x**

10.

$x + \frac{1}{y} = 12$ $\quad$ $y + \frac{1}{x} = \frac{3}{8}$

$xy + 1 = 12y$ $\quad$ $xy + 1 = \frac{3}{8}x$

$12y = \frac{3}{8}x$

$y = \frac{1}{32}x$

$x\left(\frac{1}{32}x\right) + 1 = 12(\frac{1}{32}x)$ substitution

$x^2 + 32 = 12x$ multiply by 32, simplify

$x^2 - 12x + 32 = 0$

$(x-8)(x-4) = 0$

$x = 8, x = 4$

$y = \frac{1}{32}(8), \frac{1}{32}(4)$

$y = \frac{1}{4}, \frac{1}{8}$

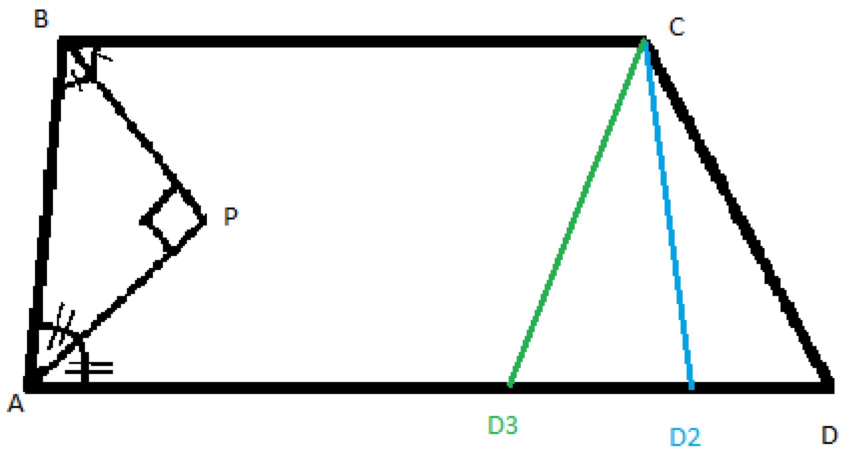
$xy = 8\left(\frac{1}{4}\right), 4(\frac{1}{8})$

$xy = 2, \frac{1}{2}$

The largest of these is 2,

So the answer is **D. 2**

11.



$\angle DAP = \angle BAP$

$\angle ABP = \angle CBP$

$\angle BPA = 90°$

$\angle ABP + \angle BAP + \angle BPA = 180°$ (They form a triangle ΔABP)

$\angle ABP + \angle BAP + 90° = 180°$ substitute 90 for BPA

$\angle ABP + \angle BAP = 90°$

$\angle CBP + \angle DAP = 90°$ substitute twice (DAP=BAP, CBP=ABP)

$\angle ABP + \angle CBP + \angle BAP + \angle DAP = 180°$

$\angle DAB + \angle ABC = 180°$ (ABP+CBP=ABC, BAP+DAP=DAB)

$\overline{BC} \parallel \overline{AD}$

$\overline{AB}$ *may or may not be parallel to* $\overline{CD}$ (D and C can be freely moved, see diagram)

So ABCD must be a trapezoid (1 set of parallel sides), and may or may not be a parallelogram or rectangle

So the answer is **A. trapezoid**

12.

We need the sum of the squares of the roots, not the roots themselves, which is good, because this equation has irrational roots.

$2x^3 - 6x^2 + 3x + 5 = 0$

$x^3 - 3x^2 + \frac{3}{2}x + \frac{5}{2} = 0$ get rid of front coefficient

$(x - a)(x - b)(x - c) = 0$ assume we have roots a, b, and c (cubic=3 roots). We just need to find $a^2 + b^2 + c^2$.

$x^3 - ax^2 - bx^2 - cx^2 + abx + acx + bcx - abc = 0$ (multiply our factors)

$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$ group like terms

We have a+b+c and ab+ac+bc.

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$

So we just need the negative of the coefficient of x^2 squared minus twice the coefficient of x

$a^2 + b^2 + c^2 = (3)^2 - 2(\frac{3}{2})$

$= 9 - 3$

$= 6$

So the answer is **B. 6**

13.

$4^{\log_2\left(2^{\frac{1}{4}}2^{\frac{1}{8}}2^{\frac{1}{16}}\dots\right)}$

$$= 4^{\log_2(2^{(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\dots)})}$$

$$= 4^{(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\dots)}$$

$$= 4^{\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\dots\right)}$$

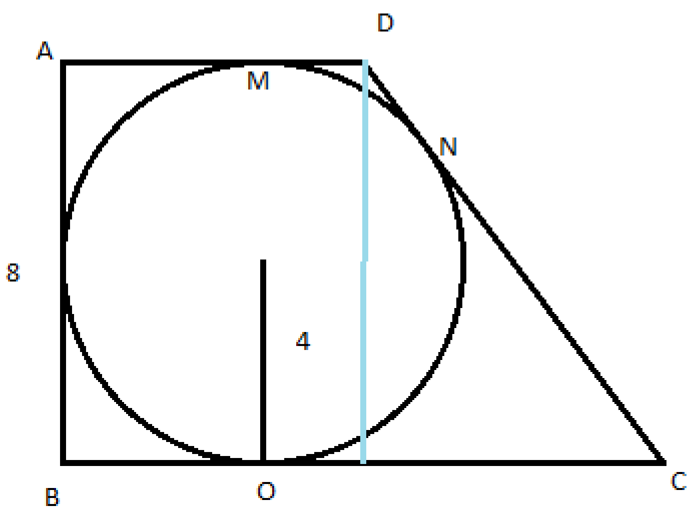
$$= 4^{\left(\frac{1}{2}\right)(1)}$$

$$= 4^{\frac{1}{2}}$$

$= 2$

So the answer is **C.2**

14



We need to find the area of this trapezoid.

$$Area\ of\ a\ trapezoid = (average\ of\ the\ bases) * height$$

$\overline{AB} = 8$ (twice the radius)

$\overline{BC} = 12$ (we are told it's 3 times the radius)

$\overline{OC} = 8$ (12 minus BO, which equals the radius)

$\overline{CN} = 8$ (OC and CN are tangent to the circle, and are thus equal)

$\overline{DN} = \overline{DM} = x$ (also tangent to circle)

$(OC - x)^2 + 8^2 = (CN + x)^2$ (drop a vertical line from D, making a right triangle)

$8^2 - 16x + x^2 + 8^2 = 8^2 + 16x + x^2$

$64 = 32x$

$x = 2$

$AD = 6$ $2 + \text{radius}$

$Area = \frac{AD+BC}{2} * AB$

$= \frac{6+12}{2} * 8$

$= 9 * 8 = 72$

So the answer is **A. 72**

15.

The only way for 6 computers to be hooked up in the manner they describe is as a ring, like so:

A→B→C
↑ ↓
D←E←F

Rotating the ring doesn't change the layout, (so only 1/6 of the orders are unique) and reverse order isn't unique (so only ½ of those, 1/12 of the total, are unique)

So the number of different ways are $\frac{6*5*4*3*2*1}{12} = \frac{720}{12} = 60$

So the answer is **D. 60**

16.

Our arithmetic sequence is $a, a + r, a + 2r, a + 3r$ …

Our geometric sequence is b, br, br^2, br^3 …

Summing gets us:

$a + b = 7$

$a + r + br = 26$

$a + 2r + br^2 = 90$

Cancelling br and br^2 doesn't look promising, so let's instead try to get rid of a and r, while incorporating all three equations.

$2a + b + 2r + br^2 = 97$ add 1st and 3rd

$-2a - 2r - 2br = -52$ 2nd times -2

$br^2 - 2br + b = 45$ add above equations. Now we can factor

$b(r-1)^2 = 3^2 * 5$ factor both sides. Since r and b are integers, the $(r-1)^2$ term has to match with 3^2, and b with 5.

$(r-1)^2 = 3^2$

$r - 1 = 3$

$r = 4$

So the answer is **B. 4**

17.

In order to figure this out, it's easiest to find the number of ways the worker can travel without crossing at all, compared to the number of ways the worker can travel total (using the complement rule).

The total number of ways the worker can travel is $13 * 12 * 11 = 1716$.

If he starts on the side with 5 rooms, there are $5 * 4 * 3 = 60$ ways he can stay on one side.

If he starts on the side with 8 rooms, there are $8 * 7 * 6 = 336$ ways

So there are 396 ways total to not cross.

1716-396=1320 ways that he crosses at least once

$\frac{1320}{1716} = \frac{10}{13}$ will then be the ratio of ways he crosses to total ways, and is therefore the probability of crossing at least once.

So the answer is **D.** $\mathbf{\frac{10}{13}}$

18

This problem is a lot easier with the knowledge that 3*37=111

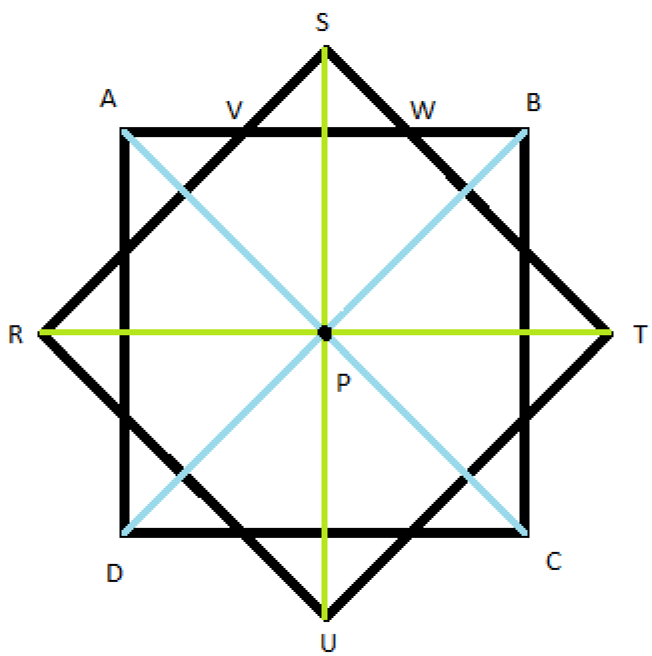
111+37=148

111+370=481

(this works with some other multiples of 37 as well)

So the answer is **B. 37**

19



The intersection of AC and BD is the center of the square, so rotating 45 degrees gets us the figure above.

To find the area, we can add the area of the square, and the 4 triangles not in the square.

The square is 10*10=100

Triangle SVW has height equal to $\frac{SU-10}{2} = \frac{1}{2}(10\sqrt{2} - 10)$

The base of SVW is twice the height (due to 45 degree angles), so is $10\sqrt{2}-10$

Area of SVW is equal to $\frac{1}{2}b*h=\frac{1}{2}(\frac{1}{2}(10\sqrt{2}-10)(10\sqrt{2}-10)=\frac{1}{4}(10\sqrt{2}-10)^2$

We have 4 triangles outside our square, all similar to SVW, so their total area is $4\left(\frac{1}{4}(10\sqrt{2}-10)^2\right)=(10\sqrt{2}-10)^2$

Adding on the square we have

$100+(10\sqrt{2}-10)^2\approx 117.16$

So the answer is **A. 117**

20

$a^2+(a+1)^2+(a+2)^2\ldots+(a+99)^2=(a+100)^2+(a+101)^2\ldots+(a+198)^2$

Let's declare a new variable b=a+99. Rewriting our equation in terms of b gets

$(b-99)^2+(b-98)^2\ldots+b^2=(b+1)^2+(b+2)^2\ldots+(b+98)^2+(b+99)^2$

If we FOIL out each term, we get

$(b^2-198b+99^2)+(b^2-196b+98^2)\ldots+(b^2-2b+1)+b^2=(b^2+2b+1)\ldots+(b^2+196b+98^2)+(b^2+198b+99^2)$

All of the b^2 terms except for the lone b^2 on the left cancel, and all of the $number^2$ terms cancel out leaving

$-198b-196b\ldots-2b+b^2=2b+4b\ldots+196b+198b$

$b^2=4(1+2+3\ldots+97+98+99)b$

$b^2=4\frac{(99)(99+1)}{2}b$ summation formula

b^2 =19800b

$b=19800$

$a=19800-99=19701$

So the answer is 19701